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ABELIAN p-GROUPS OF SYMMETRIES OF SURFACES

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Abstract. An integer $g \geq 2$ is said to be a genus of a finite group G if there is a compact Riemann surface of genus g on which G acts as a group of automorphisms. In this paper finite abelian p-groups of arbitrarily large rank, where p is an odd prime, are investigated. For certain classes of abelian p-groups the minimum reduced stable genus σ_0 of G is calculated and consequently the genus spectrum of G is completely determined for certain "extremal" abelian p-groups. Moreover for the case of $Z_p^{r_1} \oplus Z_{p^2}^{r_2}$ we will see that the genus spectrum determines the isomorphism class of the group uniquely.

1. INTRODUCTION

Let $\sum_g$ denote a closed orientable surface of genus $g \geq 2$. We consider finite groups G acting effectively on $\sum_g$ and preserving the orientation - for short, G acts on $\sum_g$ or G is a symmetry group of $\sum_g$. For each fixed g there can be only finitely many groups G that act on $\sum_g$, since by a famous result of Hurwitz [3] the order of G is bounded above by $84(g-1)$.

On the other hand, for each G there is an infinite sequence of integers $g \geq 2$ such that G acts on $\sum_g$ [4; 9]. The determination of this sequence, which is called the genus spectrum of G in [8], is referred to as the Hurwitz problem in [9]. Kulkarni [4] showed that for any given finite group G there is an integer $n_0 = n_0(G)$ such that if G acts on $\sum_g$ then $g \equiv 1(\mathrm{mod}\ n_0)$ and for all but a finite number of g such that $g \equiv 1(\mathrm{mod}\ n_0), G$ acts on $\sum_g$. If $g = 1 + n_0 g_0$ belongs to the genus spectrum of G, then $g_0 \geq 1$ is called a reduced genus of G. There is a minimum reduced genus $\mu_0 = \mu_0(G)$, and a minimum reduced stable genus; that is a smallest integer $\sigma_0 = \sigma_0(G)$ such that all g_0 with $g_0 \geq \sigma_0$ are reduced genera for G. The integers in the interval $[\mu_0, \sigma_0]$ that are not reduced genera for G form the (reduced) gap sequence of G. The genus spectrum is completely determined by n_0, μ_0, σ_0 and the (reduced) gap sequence of G. When G is a p-group with p

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odd, $n_0 = n_0(G) = p^{n-e}$ [4] where G has order p^n and exponent p^e. The integer $n - e$ is the cyclic p-deficiency of G.

The complete genus spectrum is known only for cyclic groups of prime order, elementary abelian p-groups, p-groups of cyclic p-deficiency ≤ 2 and certain other p-groups, split metacyclic groups of order pq where p and q are primes, p-groups of exponent p and p-groups of maximal class [5; 8; 13; 11].

In this paper we investigate finite abelian p-groups of arbitrarily large rank and for certain classes of them prove the following theorem:

Main Theorem. (Theorem 3.5). Let $G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e}$ $(r_e \neq 0)$. Suppose that, for $k = 0, 1, \cdots, e-1$

$$\sum_{i=k+1}^{e} (p - 1 - r_i) \geq 1.$$

Then $\sigma_0(G) = \sigma_e(p) - 1 = \frac{1}{2}[(e(p-1) - 3)p^e + 1]$.

Along the way we determine necessary and sufficient conditions for a finite abelian p-group to be a "smooth" quotient of a Fuchsian group (Theorem 3.4). This is a simple test which is useful in the computer implementation of algorithms which determine all groups G acting on a given $\sum_g$. It is immediate from the results given in [5] that two cyclic p-groups with the same genus spectrum are isomorphic. By the results given in [8] this is also true for elementary abelian p-groups. We conjecture that it is true for any pair of abelian p-groups. In this paper we present further evidence for this conjecture by showing that for an odd prime p abelian p-groups of the form $\mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2}$ having the same genus spectrum must be isomorphic (Theorem 3.8).

2. Preliminaries

It is assumed throughout that G is a p-group and that p is an odd prime. Let G have order p^n and exponent p^e, so that it has cyclic p-deficiency $n - e$ and $n_0 = n_0(G) = p^{n-e}$. It was proved in [4] that:

Theorem 2.1. *If G acts on $\sum_g$, then $g - 1 \in n_0\mathbb{N}$ and furthermore, for all but a finite number of integers g where $g \in 1 + n_0\mathbb{N}$, G acts on $\sum_g$.*

The general approach is to regard these groups of symmetries as quotient groups of Fuchsian groups and to use the following well-known result :

Theorem 2.2. *G acts on a compact surface $\sum_g$ of genus $g \geq 2$ if and only if there exist a Fuchsian group Γ and an epimorphism $\phi : \Gamma \longrightarrow G$ such that the kernel of ϕ is isomorphic to $\pi_1(\sum_g)$.*

If the kernel of ϕ is torsion-free, then ϕ is called a smooth epimorphism.

Since G is a p-group of order p^n and exponent p^e, in order for ϕ to be smooth, the periods of Γ can only be of the form p^j where $1 \leq i \leq e$. Let Γ have x_i conjugacy classes of maximal cyclic subgroups of order p^j so that Γ has a presentation of the form

$$(1)\qquad \begin{aligned} &\text{Generators : } c_{11}, c_{12}, \cdots, c_{1x_1}, c_{21}, \cdots, c_{ex_e}, a_1, b_1, \cdots, a_h, b_h; \\ &\text{Relations : } c_{ij}^{p^i} = 1 \quad \text{for} \quad i = 1, 2, \cdots, e, \;\; j = 1, 2, \cdots, x_i \\ &\prod_{k=1}^{h} [a_k, b_k] \prod_{i,j} c_{ij} = 1. \end{aligned}$$

This group has signature $(h; p^{(x_1)}, p^{2(x_2)}, \cdots, p^{e(x_e)})$, where $n^{(r)}$ indicates that the period n is repeated r times.

If $\mu(\Gamma)$ denotes the area of a fundamental region for Γ and K the kernel of ϕ, then the equation $\mu(K) = o(G)\mu(\Gamma)$ yields the Riemann - Hurwitz relation

$$(2)\qquad 2(g-1) = p^n \left[2(h-1) + \sum_{i=1}^{e} x_i \left(1 - \frac{1}{p^i} \right) \right].$$

Thus to decide whether G acts on $\sum_g$, one must determine integers $h \geq 0$ and $x_i \geq 0$ - the data $\{h; x_1, x_2, \cdots, x_e\}$ - such that (2) holds and such that there exists a smooth epimorphism ϕ from the group Γ, determined by the data, onto G. Sometimes the abbreviated notation $\{h; x_i\}$ will be used for the data.

From (2), it is immediately clear that for p odd, $g \equiv 1 \pmod{p^{n-e}}$ (see Theorem 2.1), and it remains to consider solutions of the Diophantine equation

$$(3)\qquad N = p^e h + \sum_{i=1}^{e} \frac{1}{2} x_i (p^e - p^{e-i}).$$

Let $\Omega_e = \Omega_e(p)$ denote all the solutions N of (3) for which $h \geq 0$ and $x_i \geq 0$ for all i. We know from [5] that:

Theorem 2.3. *If $2N = a_0 + a_1 p + a_2 p^2 + \cdots + a_e p^e$ is a truncated p-adic expansion of $2N$, where $0 \leq a_i < p$ for $i = 0, 1, \cdots, e-1$ and $a_e \geq 0$, then $\Omega_e = \{N \in \mathbb{N} \mid S_e(2N) = \sum_{k=0}^{e} a_k \geq (e-i)(p-1)\}$. Here a_i is the first nonzero coefficient in the expansion of $2N$.*

and

Corollary 2.4. *Let $\sigma_e(p)$ denote the minimum stable solution in $\Omega_e(p)$; that is, $\sigma_e(p)$ is minimal with the property that all $N \geq \sigma_e(p)$ lie in $\Omega_e(p)$. Then $\sigma_e(p) = \frac{1}{2}[(e(p-1)-3)p^e + 3]$.*

Definition 2.5. (1) For each N with $1 \leq N \leq e$, define

$$\Lambda^N(G) =< x \in G \mid x^{p^N} = 1 > .$$

(2) G is said to have property M_N if all elements of order $\geq p^N$ lie in $G \setminus G'\Lambda^{N-1}(G)$.

(3) The group G is said to have the maximal exponent property (MEP) if G has exponent p^e and property M_e.

Then we have the following results from [8]:

Lemma 2.6. *Let G act on $\sum_g$ $(g \geq 2)$ with corresponding Fuchsian group Γ, whose associated data is $\{h; x_1, x_2, \cdots, x_N\}$ for some $N \leq e$. If G has property M_N and $x_N \neq 0$, then $x_N \geq 2$.*

Theorem 2.7. *Let G have exponent p^e. If G has MEP, then* $\sigma_0(G) \geq \sigma_e(p) - 1$.

3. Abelian p-Groups of Symmetries of Surfaces

Throughout this section we consider abelian p-groups of arbitrary rank where p is an odd prime. Note that we continue our use of the word *rank* in the sense of minimal number of generators.

In this section we first obtain simple necessary and sufficient conditions on the signature of a Fuchsian group so that there will be a smooth homomorphism onto an arbitrary given abelian p-group. We then use these conditions,together with a key computational result, Lemma 3.6, to prove our main result, Theorem 3.5.

Throughout this section, let

$$G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e} \quad (r_e \geq 1) \tag{4}$$

be an abelian p-group of exponent p^e. Fix canonical generators

$$\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1r_1}, \alpha_{21}, \cdots, \alpha_{er_e}$$

for G, and denote α_{er_e} by α.

Let Γ be a Fuchsian group with data $\{h; x_1, x_2, \cdots, x_e\}$. If all $x_i = 0$, define $N = 0$. Otherwise, put N equal to the largest i such that $x_i \neq 0$. Clearly $N \leq e$. The abelianization A of Γ can be written as

$$A \cong \mathbb{Z}^{2h} \oplus \mathbb{Z}_p^{y_1} \oplus \cdots \oplus \mathbb{Z}_{p^N}^{y_N}.$$

Lemma 2.6 says that $x_N \geq 2$ when $N > 0$, and we then have $y_i = x_i$ for $i < N$ and $y_N = x_N - 1 \geq 1$. Fix canonical generators for A:

$$A_1, B_1, \cdots, A_h, B_h, C_{11}, \cdots, C_{1y_1}, \cdots, C_{Ny_N}.$$

Then we have :

Theorem 3.1. *There exists a smooth epimorphism $\varphi : \Gamma \longrightarrow G$ if and only if there exists an epimorphism $\phi : A \longrightarrow G$ such that $\phi(C_{ij})$ has order p^i and $\phi\left(\prod_{i,j} C_{ij}\right)$ has order p^N. Such an epimorphism ϕ we will also call a smooth epimorphism.*

Lemma 3.2. *Let $G \cong Z_p^{r_1} \oplus Z_{p^2}^{r_2} \oplus \cdots \oplus Z_{p^e}^{r_e}$, $(r_e \neq 0)$ be an abelian p-group. If there exists a Fuchsian group Γ with signature $(h; p^{(y_1)}, p^{2(y_2)}, \cdots, p^{N(y_{N+1})})$ where $0 \leq N \leq e$ and a smooth epimorphism $\varphi : \Gamma \to G$ then the following inequalities are satisfied :*

$$\begin{aligned} 2h + y_1 + y_2 \cdots + y_N &\geq r_1 + r_2 + \cdots + r_e \\ 2h + y_2 + \cdots + y_N &\geq r_2 + \cdots + r_e \\ \vdots \quad &\quad \vdots \\ 2h + y_N &\geq r_N + \cdots + r_e \\ 2h &\geq r_{N+1} + \cdots + r_e \ \text{ if } \ N < e \ . \end{aligned}$$

Proof. First we show that if A is a finitely generated abelian p-group, B is a finite abelian p-group and $\psi : A \to B$ is an epimorphism then ψ induces an epimorphism

$$\psi^{(k)} : A/\Lambda^k(A) \to B/\Lambda^k(B)$$

for all k, where $\Lambda^k(A) =< x | x \in A \text{ and } x^{p^k} = 1 >$:

Let $\pi_1 : A \to A/\Lambda^k(A)$ and $\pi_2 : B \to B/\Lambda^k(B)$ be the natural epimorphisms. Then for every k there exists a mapping

$$\psi^{(k)} : A/\Lambda^k(A) \to B/\Lambda^k(B)$$

such that $\psi^{(k)}\pi_1 = \pi_2\psi$. In other words

$$\psi^{(k)}(a\Lambda^k(A)) = \psi(a)\Lambda^k(B) \text{ for every } a\Lambda^k(A) \in A/\Lambda^k(A).$$

Using the fact that ψ is an epimorphism, it can be seen that $\psi^{(k)}$ is an epimorphism for all k.

Now we are ready to prove the Lemma :

Denote Γ/Γ' by A. Then

$$A \cong Z^{2h} \oplus Z_p^{y_1} \oplus \cdots \oplus Z_{p^N}^{y_N}$$

is a finitely generated abelian p-group. Let $\pi : \Gamma \to A$ be the natural epimorphism. Then there exists a mapping $\phi : A \to G$ such that $\phi\pi = \varphi$. Then ϕ is an

epimorphism and if we take $\psi = \phi$ in the result we proved previously it induces an epimorphism

$$\phi^{(k)} : A/\Lambda^k(A) \to G/\Lambda^k(G) \text{ for } k = 1, 2, \cdots, N-1.$$

Therefore $r(A) \geq r(G)$ and

$$r(A/\Lambda^k(A)) \geq r(G/\Lambda^k(G)) \text{ for } k = 1, 2, \cdots, N-1$$

where r denotes the rank of the given group. Here

$$\Lambda^k(A) \cong {Z_p}^{y_1} \oplus {Z_{p^2}}^{y_2} \oplus \cdots \oplus {Z_{p^{k-1}}}^{y_{k-1}} \oplus {Z_{p^k}}^{y_k+y_{k+1}+\cdots+y_N}$$

$$A/\Lambda^k(A) \cong Z^{2h} \oplus {Z_p}^{y_{k+1}} \oplus {Z_{p^2}}^{y_{k+2}} \oplus \cdots \oplus {Z_{p^{N-k}}}^{y_N}$$

$$\Lambda^k(G) \cong {Z_p}^{r_1} \oplus \cdots \oplus {Z_{p^k}}^{r_k+r_{k+1}+\cdots+r_e}$$

and finally

$$G/\Lambda^k(G) \cong {Z_p}^{r_{k+1}} \oplus {Z_{p^2}}^{r_{k+2}} \oplus \cdots \oplus {Z_{p^{e-k}}}^{r_e}.$$

Thus we have the following inequalities :

$$2h + y_{k+1} + \cdots + y_N \geq r_{k+1} + \cdots + r_e \text{ for } k = 0, 1, \cdots, N-1.$$

We have

$$\Lambda^N(A) \cong Z_p^{y_1} \oplus \cdots \oplus Z_{p^N}^{y_N},$$

$$\Lambda^N(G) \cong Z_p^{r_1} \oplus \cdots \oplus Z_{p^N}^{r_N+r_{N+1}+\cdots+r_e},$$

$$A/\Lambda^N(A) \cong Z^{2h},$$

$$G/\Lambda^N(G) \cong Z_p^{r_{N+1}} \oplus \cdots \oplus Z_{p^{e-N}}^{r_e}.$$

Hence $2h \geq r_{N+1}+\cdots+r_e$ if $N<e$ which completes the proof of the Lemma. ■

Suppose $\phi : A \longrightarrow G$ is a smooth epimorphism. Since $\phi(\Lambda^k(A)) \subseteq \Lambda^k(G)$ ϕ induces epimorphisms

$$\phi^{(k)} : A/\Lambda^k(A) \longrightarrow G/\Lambda^k(G)$$

for each k. Observe that

$$A/\Lambda^k(A) \cong \mathbb{Z}^{2h} \oplus \mathbb{Z}_p^{y_{k+1}} \oplus \cdots \oplus \mathbb{Z}_{p^{N-k}}^{y_N}$$

for $1 \leq k < N$ and is isomorphic to $\mathbb{Z}^{2h}$ for $k \geq N$, while

$$G/\Lambda^k(G) \cong \mathbb{Z}_p^{r_{k+1}} \oplus \cdots \oplus \mathbb{Z}_{p^{e-k}}^{r_e}$$

for $1 \leq k \leq e$. By comparing ranks it follows that

$$\begin{aligned} 2h + \sum_{i=k+1}^{N} y_i &\geq \sum_{i=k+1}^{e} r_i \quad \text{for} \quad k = 0, 1, 2, \cdots, N-1 \\ 2h &\geq \sum_{i=N+1}^{e} r_i \quad \text{if} \quad N < e. \end{aligned} \tag{5}$$

We now aim to show that these conditions are sufficient for the existence of a smooth epimorphism $\phi : A \longrightarrow G$. For this purpose we introduce the following notation :

Let G be as in (4). If the sequence $\{h; N; y_1, \cdots, y_N\}$ of nonnegative integers with $0 \leq N \leq e$, $y_N \neq 0$ satisfies the inequalities (5) then it is called a *G-data.*

Associated to any set of G-data, there is a *G-data group* A, the finitely-generated abelian group

$$A = \mathbb{Z}^{2h} \oplus \mathbb{Z}_p^{y_1} \oplus \cdots \oplus \mathbb{Z}_{p^N}^{y_N}$$

Theorem 3.3. *Given any finite abelian p-group G, any G-data group A, there exists a smooth epimorphism $\phi : A \longrightarrow G$.*

Proof. Suppose the G-data satisfies (5) with $N = e$. For $i = 1, 2, \cdots, e-1$, let $X_i = \{C_{ij} : 1 \leq j \leq y_i\}$ and let

$$X_e = \{A_1, B_1, \cdots, A_h, B_h, C_{ej} : 1 \leq j \leq y_e\}.$$

Choose $Y_e \subseteq X_e$ such that $C_{ey_e} \in Y_e$ and $\mid Y_e \mid = r_e$. Let $Z_e = X_e \setminus Y_e$. Now choose Y_i inductively for $i = e-1, \cdots, 1$ as follows: $Y_i \subseteq X_i \cup Z_{i+1}$ such that $\mid Y_i \mid = r_i$ and let $Z_i = (X_i \cup Z_{i+1}) \setminus Y_i$. By the given inequalities this is always possible.

Let $\phi(C_{ey_e}) = \alpha^t$ with $(t, p) = 1$ and let ϕ map the remaining generators in Y_e onto $\{\alpha_{e1}, \cdots, \alpha_{er_e-1}\}$. Let the image of each element of Y_i be $\alpha^{\lambda_{i1}}\alpha_{i1}, \cdots, \alpha^{\lambda_{ir_i}}\alpha_{ir_i}$ where the λ_{ij} are chosen such that, if $\gamma \in Y_i$ has finite order then $o(\gamma) = o(\phi(\gamma))$. If the order is infinite then we may choose the corresponding λ_{ij} to be zero. Finally map each element of Z_1 onto α^{λ_i} where λ_i is chosen such that the orders of finite elements are preserved. Note that $\phi\left(\prod_{i,j} C_{ij}\right)$ will be $\alpha_{11} \cdots \alpha_{er_e-1}\alpha^{\mu}$ where $\mu = t + \sum \lambda_i + \sum \lambda_{ij}$. If $r_e > 1$, then this element automatically has order p^e. If $r_e = 1$, we choose t such that $(\mu, p) = 1$ and so that ϕ is smooth.

If the G-data satisfies (5) for $N < e$ we use a similar argument. Let $X_i = \{C_{ij} : 1 \leq j \leq y_i\}$ for $i = 1, 2, \cdots, N$ and $X_{N+1} = \{A_1, B_1, \cdots, A_h, B_h\}$.

Choose $Y_{N+1} \subseteq X_{N+1}$ such that $\mid Y_{N+1} \mid = r_{N+1} + \cdots + r_e$ and let $Z_{N+1} = X_{N+1} \setminus Y_{N+1}$. Now choose Y_i and Z_i inductively as above.

Then ϕ will be chosen to map all elements of Y_{N+1} onto $\alpha_{N+11}, \cdots, \alpha_{er_e}$ $(= \alpha)$, and be defined on the remaining generators as above so that it is smooth. ■

Combining Lemma 2.6 with the inequalities (5) and Theorem 3.3, we obtain :

Theorem 3.4. *Let $G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e}$ and let Γ be a Fuchsian group with signature $(h; p^{(x_1)}, p^{2(x_2)}, \cdots, p^{e(x_e)})$. Let $N = 0$ if all $x_i = 0$, otherwise let N be the largest i with $x_i \neq 0$. Then, there is a smooth epimorphism $\phi : \Gamma \longrightarrow G$ if and only if*

$$N = 0 \ \text{ or } \ x_N \ \geq \ 2,$$

and

$$2h - 1 + \sum_{i=k+1}^{N} x_i \geq \sum_{i=k+1}^{e} r_i \quad \text{for} \quad k = 0, 1, 2, \cdots, N-1, \tag{6}$$

and

$$2h \ \geq \ \sum_{i=N+1}^{e} r_i \quad \text{if} \quad N < e.$$

Proof of Theorem 3.1. Note that this Theorem is another way of expressing Theorem 3.4 and the proof follows from Lemma 3.2, the remarks preceeding it and Theorem 3.3.

Note that, it follows that, g_0 is a reduced genus for G if and only if g_0 satisfies the equation

$$g_0 + p^e = p^e h + \sum_{i=1}^{e} \frac{1}{2} x_i (p^e - p^{e-i}) \tag{7}$$

where the data $\{h; x_i\}$ satisfies at least one of the conditions (6) for $N = 0, 1, 2, \cdots, e$.

Recall that

$$\sigma_0(G) \geq \sigma_e(p) - 1 = \frac{1}{2}[(e(p-1) - 3)p^e + 1].$$

We are now ready to state our main result:

Theorem 3.5. *Let $G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e}$ $(r_e \neq 0)$. Suppose that for $k = 0, 1, \cdots, e-1$, we have*

$$\sum_{i=k+1}^{e} (p - 1 - r_i) \geq 1.$$

Then $\sigma_0(G) = \sigma_e(p) - 1$.

Proof. By Theorem 2.7, we have $\sigma_0(G) \geq \sigma_e(p) - 1$. For the reverse inequality, we must show that g_0 is a reduced genus for G whenever $g_0 \geq \sigma_e(p) - 1$. We will use the following Lemma:

Lemma 3.6. *Suppose that $M \geq \frac{1}{2}[(e(p-1)-1)p^e + 1]$ and N is the smallest non-negative integer for which p^{e-N} divides M. Then there exists a solution to*

$$2M = 2p^e h + \sum_{i=1}^{e} x_i(p^e - p^{e-i})$$

where the data $\{h; x_i\}$ satisfy the conditions

- $x_i \geq 0, \;\; x_i = 0 \;\; for \;\; i > N, \;\; and \;\; if \;\; N > 0 \;\; then \;\; x_N \geq 2,$
- $2h + x_{k+1} + \cdots + x_N \geq (e-k)(p-1) \;\; for \;\; k = 0, 1, \cdots, N-1$
- $2h \geq (e-N)(p-1).$

Proof. Before proving Lemma 3.6, we will complete the proof of Theorem 3.5. Write $M = g_0 + p^e$, so that using Corollary 2.4, the condition $g_0 \geq \sigma_e(p) - 1$ becomes $M \geq \frac{1}{2}((e(p-1)-1)p^e + 1)$. Choose $N \geq 0$ minimal such that p^{e-N} divides M, and apply Lemma 3.6. Using the hypothesis of Theorem 3.5, we have for $0 \leq k \leq N-1$ that

$$2h - 1 + \sum_{i=k+1}^{N} x_i \geq (e-k)(p-1) - 1 \geq \sum_{i=k+1}^{e} r_i,$$

and similarly if $N < e$ then $2h \geq (e-N)(p-1) \geq \sum_{i=N+1}^{e} r_i$. Lemma 3.6 also provides that $N = 0$ or $x_N \geq 2$, so Theorem 3.4 applies to show that g_0 is a reduced genus for G.

Let the statement of the Lemma be denoted by $P(e, N)$ where $0 \leq N \leq e$ and $e \geq 1$. The result is proved by double induction. Consider the case $P(e, 0)$. Then $2M = 2zp^e$, and choosing $h = z$, all $x_i = 0, \;\; P(e, 0)$ holds.

Now consider $P(1, 1)$. Then $2M = z_0 + z_1 p$ where $0 < z_0 < p$, and the assumption on M forces $z_1 \geq p - 2$. If $z_0 \neq p - 1$, choose $x_1 = p - z_0$ and $2h = z_0 + z_1 + 1 - p$. If $z_0 = p - 1$, let $x_1 = p + 1$ and $2h = z_1 + 1 - p$.

Now consider $P(e, N)$ where $N < e$. So $2M = 2M' p^{e-N}$. Let $2M'' = 2M' - (e-N)(p-1)p^N$. So by the inductive assumption on $P(N, N)$, there is a solution to the equation for M'' with $x''_N \geq 2$ and $h'' \geq 0$ and

$$2h'' + x''_{k+1} + \cdots + x''_N \geq (N-k)(p-1)$$

for $0 \leq k \leq N-1$. But then, setting $x_i = x''_i$ for $i = 1, 2, \cdots, N$, $x_{N+1} = \cdots = x_e = 0$ and $2h = 2h'' + (e-N)(p-1)$, it follows that $P(e, N)$ holds.

Now consider $P(e, e)$. Suppose $2M \equiv z_0 (\mathrm{mod}\ p)$ with $0 < z_0 < p$. If $z_0 \neq p - 1$, let

$$2M'p = 2M - (p - z_0)(p^e - 1) - (z_0 - 1)p^e - tp$$

where $t = 0, 1$ according as z_0 is odd or even. Then $P(e-1, N')$ applies to M' for some $0 \leq N' \leq e-1$. The data for the solution satisfy $x'_{N'} \geq 2$, $x'_{N'+1} = \cdots = x'_{e-1} = 0$,

$$2h' + x'_{k+1} + \cdots + x'_{N'} \geq (e - 1 - k)(p-1)$$

for $0 \leq k < N'$ and $2h' \geq (e - 1 - N')(p-1)$. Setting $x_i = x'_i$ for $i = 1, 2, \cdots, e-1$ and

$$(h, x_e) = \begin{cases} (h' + \frac{1}{2}(z_0 - 1), p - z_0) & \text{if } z_0 \text{ is odd} \\ \\ (h' + \frac{z_0}{2}, p - z_0 - 1) & \text{if } z_0 \text{ is even} \end{cases}.$$

It follows that $P(e, e)$ holds in this case.

Now suppose $z_0 = p - 1$. Let

$$2M'p = 2M - (p+1)(p^e - 1) + 2(p^e - p).$$

Then $P(e-1, N')$ applies to M' for some $0 \leq N' \leq e-1$. If $N' = e-1$ then $2M' = 2h'p^{e-1} + \sum_{i=1}^{e-1} x'_i(p^{e-1} - p^{e-1-i})$ with $x'_{e-1} \geq 2$ and

$$2h' + x'_{k+1} + \cdots + x'_{e-1} \geq \quad (e - 1 - k)(p-1)$$

for $0 \leq k < e-1$. Then choosing $x_e = p+1$, $x_{e-1} = x'_{e-1} - 2$ and $x_i = x'_i$ for $i \leq e-2$, and $h = h'$, we see that $P(e, e)$ holds in this case.

It remains to consider the case $z_0 = p - 1$ and $N' < e - 1$, in which case $2M \equiv p - 1 (\mathrm{mod}\ p^s)$ for some $s \geq 2$. Let s be the highest such power. Let

$$\begin{aligned} 2M'p^s &= 2M - (p+1)(p^e - 1) - (p-2)(p^e - p) - (p-1)\sum_{i=2}^{s-1}(p^e - p^i) \\ &\quad +(p^e - p^s) \\ &= 2M - (p-1) - s(p-1)p^e. \end{aligned}$$

So $P(e-s, e-s)$ applies to M' and we have a solution to

$$2M' = 2h'p^{e-s} + \sum_{i=1}^{e-s} x'_i(p^{e-s} - p^{e-s-i})$$

with $x'_{e-s} \geq 2$ and $h' \geq 0$ and

$$2h' + x'_{k+1} + \cdots + x'_{e-s} \geq (e - s - k)(p-1)$$

for $0 \leq k < e-s$. Now choosing $x_e = p+1, x_{e-1} = p-2, \quad x_{e-2} = \cdots = x_{e-s+1} = p-1, \quad x_{e-s} = x'_{e-s} - 1, \quad x_i = x'_i$ for $i < e-s$ and $h = h'$ again shows that $P(e,e)$ holds. ∎

If we consider the "extremal" groups given by Theorem 3.5, we have:

Corollary 3.7. *Let* $G \cong \mathbb{Z}_p^{p-1} \oplus \mathbb{Z}_{p^2}^{p-1} \oplus \cdots \oplus \mathbb{Z}_{p^{e-1}}^{p-1} \oplus \mathbb{Z}_{p^e}^{p-2}$. *Then* $\mu_0(G) = \sigma_0(G) = \sigma_e(p) - 1$.

Proof. The computation of $\mu_0(G)$ follows from [6]. ∎

Finally we have the following uniqueness theorem:

Theorem 3.8. *Let* $G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2}, \quad H \cong \mathbb{Z}_p^{s_1} \oplus \mathbb{Z}_{p^2}^{s_2}$ *where* p *is an odd prime. If the genus spectrum of* G *and the genus spectrum of* H *are the same then* $G \cong H$.

Proof. If G and H have the same genus spectrum, then they must have the same cyclic p-deficiency; also $\mu_0(G) = \mu_0(H), \quad \sigma_0(G) = \sigma_0(H)$ and their reduced gap sequences are equal. In this proof we only make use of the fact that the cyclic p-deficiencies are equal and that $\mu_0(G) = \mu_0(H)$. The minumum reduced genera $\mu_0(G)$ and $\mu_0(H)$ can be determined by using the results given in [6]. The result follows by showing that $r_1 = s_1$ and $r_2 = s_2$. ∎

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